

# Singularities of the Casimir Energy for Quantum Field Theories with Lifshitz Dimensions

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## Abstract

We study the singularities that the Casimir energy of a scalar field in spacetimes with Lifshitz dimensions exhibits, and provide expressions of the energy in terms of multidimensional zeta functions for the massive and massless case. We found that when extra dimensions are considered, the critical exponents of the Lifshitz dimensions affect drastically the Casimir energy, introducing singularities that are absent in the non-Lifshitz case. When the four dimensional case is considered, the Casimir energy has less singularities compared to the non-Lifshitz case. In addition for specific values of the critical exponents, the Casimir energy is finite.

## Introduction

During the last decades, the Casimir effect has played a prominent role in various research areas of theoretical physics [1–70]. The first theoretical study was done by Casimir [1], in 1948, who predicted an attractive force between two neutral perfectly conducting parallel plates. Some decades later was realized that the Casimir effect is a manifestation of the vacuum corresponding to a quantum field theory. Owing to the vacuum fluctuations of the electromagnetic quantum field, the parallel plates attract each other. Moreover, the boundaries alter the quantum field boundary conditions and as a result the plates interact. The geometry of the boundaries have a strong effect on the Casimir energy and Casimir force. The calculations of the Casimir energy and the corresponding force were generalized to include other quantum fields such as fermions, bosons and scalar fields making the Casimir effect study an important ingredient of many theoretical physics subjects such as string theory, cosmology e.t.c. The technological applications of the Casimir force are of invaluable importance, for instance in nanotubes, nano-devices and generally in microelectronic engineering [71]. Indeed an attractive or repulsive Casimir force can lead to the instability or even destruction of such a micro-device. Hence, studying various

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geometrical and material configurations will enable us to have control over the Casimir force.

The Casimir effect was verified experimentally [72, 73] rendering the physics of such studies very valuable due to the theoretical outcomes of these studies. The Casimir energy studies have been done for curved spacetimes, for topologically non trivial backgrounds and for various geometrical configurations and physical setups [1–70]. The applications of such calculations are numerous, constraining even cosmological models. Moreover, various models are put in test, for example the size and the shape of extra dimensions are constrained from Casimir energy calculations [74, 75]. In view of these applications, every consistent quantum field theory is severely constrained by Casimir energy and Casimir force measurements.

Lifshitz type quantum field theories [77–99] serve as Lorentz violating field theories with remarkable properties. These theories have their origin in condensed matter physics [100] where a Lifshitz critical point is defined as a point in a phase diagram where three phases of a condensed matter system meet. Condensed matter systems exhibiting a Lifshitz point have an intrinsic space anisotropy, which is quantified in terms of the existence of two different correlation lengths in the anisotropic space dimensions. Lifshitz quantum field theories have been studied in flat and curved background and, although being Lorentz violating theories, the renormalization of the various loop integrals is improved, with the last being the most appealing attribute of these gauge theories. In addition, the class of the renormalizable interactions is sufficiently enriched, since the Lifshitz type operators that appear in a Lagrangian are higher derivatives of the quantum fields with mass dimensions. Another appealing feature is that within the Lifshitz theoretical framework, dynamical mass generation naturally occurs, as a consequence of the dimensionfull couplings of Lifshitz operators. We shall study a scalar field, massless or massive, in the context of Lifshitz type dimensions, but in two different situations.

Firstly the anisotropy of 4-dimensional spacetime will be intact and we shall assume that spacetime has extra dimensions, which are anisotropic, in the Lifshitz way. In this framework the four dimensional Minkowski spacetime Lorentz invariance is protected, while each extra dimension is anisotropic. However this anisotropy has a radical impact on the field theory Lagrangian, since it allows a quite large number of higher order derivatives of the fields. Nevertheless, when the theory is assumed to "sit" on a Lifshitz point, the only derivative terms that survive are the higher order ones, since the lower order derivatives at the Lifshitz point become irrelevant. This construction is very similar to the original condensed matter construction and additionally is more physically appealing since it doesn't affect the four dimensional Lorentz invariance. Secondly we shall assume that our Minkowski spacetime has an anisotropy between space and time, with the last being very commonly used in these type of quantum field theories.

Since the Casimir energy has proved to be a very important property of any quantum field theory, we shall compute the Casimir energy of a scalar field, firstly in four dimensional spacetime with Lifshitz anisotropic extra dimensions and then the Casimir energy in four dimensions with anisotropic scaling between space and time. We extend the last framework in the case which each space dimension is anisotropic. What we are mainly interested in, is the singularity structure of the Casimir energy in all the aforementioned cases. We

want to explore if the introduction of a Lifshitz anisotropy in space introduces any new singularities or if it removes any singular terms. Since the Casimir energy is a benchmark for any viable quantum field theory, such a study is very important for the intrinsic validity of any quantum field theory.

This paper is organized as follows: In section 1 we calculate the Casimir energy of a massless scalar field in a Minkowski spacetime background with a Lifshitz extra dimension and exploit how the singularities of the Casimir energy are affected by the anisotropic scaling of the extra dimension. In section 2 we present the singularities of the massless scalar field Casimir energy, in Minkowski spacetime with two anisotropic extra dimensions. In section 3 we compute the Casimir energy of a massive scalar field in Minkowski spacetime with one extra dimension. Formulas for the massless multidimensional extra space Casimir energy are provided in section 4. In section 5 we compute the Casimir energy for Minkowski spacetime with anisotropic scaling for space and time, with the massless scalar field confined in a box, and satisfying Dirichlet boundary conditions on the boundaries. We also consider the case where the three space dimensions are anisotropic. The conclusions with a discussion on the results follows in the end of this paper.

## 1 Casimir Energy for Massless Scalar Field $M_4 \times S_{z_1}^1$

In this section we compute the Casimir energy for a scalar quantum field in  $M_4 \times S_{z_1}^1$  spacetime, with  $M_4$  denoting the four dimensional Minkowski spacetime, while  $S_{z_1}^1$  is the extra dimension with anisotropic scaling. This anisotropy is quantified in terms of the dynamical critical exponent  $z_1$ , and this means that the scaling of the dimensions is of the form:

$$x_\mu \rightarrow bx_\mu, \quad y_1 \rightarrow b^{z_1}y_1 \quad (1)$$

In the above equation,  $\mu = 0, 1, 2, 3$  and  $x_\mu$  denotes the Minkowski spacetime coordinates, while  $y_1$  denotes the coordinate describing the extra dimension, which is a circle. The action of the free massless scalar field in  $M_4 \times S_{z_1}^1$  is equal to,

$$\mathcal{S} = \int dt d^{D-1}p \int_0^{2\pi R_1} dy_1 \Phi^*(x_\mu, y_1) \left( \partial^\mu \partial_\mu - q_1^{2(z_1-1)} (-\partial_{y_1}^2)^{z_1} \right) \Phi(x_\mu, y_1) \quad (2)$$

Note that in the above action we defined the quantum field theory on the Lifshitz critical point  $z_1$ , thus only the highest order derivatives appear. The mass dimensions of the coordinates are equal to:

$$[x_\mu] = -1, \quad [y_1] = -z_1 \quad (3)$$

and this explains the necessity of the dimensionfull mass parameter  $q$ . We assume that the scalar field satisfies periodic boundary conditions in the Lifshitz extra dimension,  $\Phi(x_\mu, 0) = \Phi(x_\mu, 2\pi R_1)$ , with  $R_1$  the radius of the extra dimension. We shall use the dimensional regularization technique in order to calculate the Casimir energy. Therefore we suppose that  $D$  is the total ordinary space dimensionality, hence the Casimir energy

reads,

$$\mathcal{E} = \int d^{D-1}p \sum_{n_1=-\infty}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + q_1^{2(z_1-1)} \left( \frac{n_1\pi}{R_1} \right)^{2z_1} \right]^{-s} \quad (4)$$

It is crucial in the end to replace  $s = -1/2$  and  $D = 4$ . Notice that the integration is performed for the non-compact space dimensions, which are  $D - 1$ . In order to distinguish between contributions to the energy, coming from the extra dimensions and contributions coming from the  $D - 1$  dimensions, we rewrite the Casimir energy in the form:

$$\mathcal{E} = \mathcal{E}^{D-1} + \mathcal{E}_{z_1} \quad (5)$$

The term  $\mathcal{E}^{D-1}$  is the  $D - 1$ -dimensional contribution to the Casimir energy, and  $\mathcal{E}_{z_1}$  denotes the extra dimensional contribution to the Casimir energy. These two are equal to:

$$\begin{aligned} \mathcal{E}^{D-1} &= \int d^{D-1}p \left[ \sum_{k=1}^{D-1} p_k^2 \right]^{-s} \\ \mathcal{E}_{z_1} &= \int d^{D-1}p \sum_{n_1=-\infty}^{\infty'} \left[ \sum_{k=1}^{D-1} p_k^2 + q_1^{2(z_1-1)} \left( \frac{n_1\pi}{R_1} \right)^{2z_1} \right]^{-s} \end{aligned} \quad (6)$$

where the prime in the summation indicates the omission from the sum of the  $n_1 = 0$  term. We shall be mostly interested in the contribution coming from the extra dimension, and particularly we shall exploit the singularities of the Casimir energy and the effect of the critical exponent  $z_1$  on these singularities, in comparison to the non-Lifshitz case ( $z_1 = 1$ ). Upon integrating over the continuous dimensions using the formula,

$$\int dk^{D-1} \frac{1}{(k^2 + A)^s} = \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \frac{1}{A^{s - \frac{D-1}{2}}} \quad (7)$$

the extra dimensional contribution to the energy is,

$$\mathcal{E}_{z_1} = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_1=-\infty}^{\infty'} \left[ q_1^{2(z_1-1)} \left( \frac{n_1\pi}{R_1} \right)^{2z_1} \right]^{-(s - \frac{D-1}{2})}. \quad (8)$$

Using the zeta function regularization, with the zeta function being equal to:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (9)$$

the Casimir energy (8) can be written in terms of the zeta function, namely

$$\mathcal{E}_{z_1} = \frac{2}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} q_1^{-2(z_1-1)(s - \frac{D-1}{2})} \left( \frac{\pi}{R_1} \right)^{-2z_1(s - \frac{D-1}{2})} \zeta\left(2z_1\left(s - \frac{D-1}{2}\right)\right) \quad (10)$$

Obviously, the zeta function is singular when its argument, namely  $2z_1(s - \frac{D-1}{2})$ , is equal to one, which cannot be true for any value of the critical exponent  $z_1$ . Recall that we take  $D = 4$  and  $s = -1/2$  in order to recover the Minkowski space with an extra dimension case. The term  $\mathcal{E}^{D-1}$  which does not depend on the extra dimensions contains very well known singularities but we focus on the extra dimensions dependent terms of the Casimir energy, as we already mentioned, and this is what we do hereafter. The case  $z_1 = 1$  corresponds to the non-Lifshitz field theory case, in which case,  $\mathcal{E}_{z_1}$  is finite. It is easy to verify that any value of  $z_1$ , with  $z_1 \geq 2$  renders  $\mathcal{E}_{z_1}$  infinite, with the singularity being a pole of the gamma function, that is:

$$\mathcal{E}_{z_1} \sim \Gamma(-2)\zeta(-4z_1) \quad (11)$$

Therefore, we conclude that the Casimir energy is infinite and new regularization terms are probably needed to render the Casimir energy finite, a fact that does not happen for the non-Lifshitz field theories. Hence the scalar field theory defined on the critical point  $z_1$ , in the spacetime  $M_4 \times S_{z_1}^1$ , with  $z_1 \geq 2$  introduces singularities that do not exist in the non-Lifshitz case. A final notice regarding the singularities. For the case  $z_1 = 1$ , the product of the gamma function with the zeta function satisfy the reflection formula

$$\Gamma(\frac{z}{2})\zeta(z) = \pi^{z-1/2}\Gamma(\frac{1-z}{2})\zeta(1-z) \quad (12)$$

which renders the extra dimensions Casimir energy contribution finite, with the last being singular for  $z_1 \geq 2$ .

## 2 Casimir Energy for Massless Scalar Field $M_4 \times S_{z_1}^1 \times S_{z_2}^1$

We now calculate the Casimir energy for a spacetime of the form,  $M_4 \times S_{z_1}^1 \times S_{z_2}^1$ , with the extra dimensions being Lifshitz type dimensions. The dimensions have scalings that this time are of the form,

$$x_\mu \rightarrow bx_\mu, \quad y_1 \rightarrow b^{z_1}y_1, \quad y_2 \rightarrow b^{z_2}y_2 \quad (13)$$

The action of the free massless scalar field with two periodic Lifshitz extra dimensions is equal to,

$$\begin{aligned} \mathcal{S} = & \int dt d^{D-1}p \int_0^{2\pi R_1} \int_0^{2\pi R_2} dy_1 dy_2 \\ & \times \Phi^*(x_\mu, y_i) \left( \partial^\mu \partial_\mu - q_1^{2(z_1-1)} (-\partial_{y_1}^2)^{z_1} - q_2^{2(z_2-1)} (-\partial_{y_2}^2)^{z_2} \right) \Phi(x_\mu, y_i) \end{aligned} \quad (14)$$

with  $i = 1, 2$ ,  $\mu = 0, 1, 2, 3$  and  $q_1, q_2$  mass parameters in order the dimensionality of the derivative terms are correct. In this case, the dimensionality of the coordinates is:

$$[x_\mu] = -1, \quad [y_i] = -z_i \quad (15)$$

We deploy periodic boundary conditions for the scalar field in the extra dimensions:

$$\Phi(x_\mu, 0, y_2) = \Phi(x_\mu, 2\pi R_1, y_2), \quad \Phi(x_\mu, y_1, 0) = \Phi(x_\mu, y_1, 2\pi R_2) \quad (16)$$

with  $R_1, R_2$  denoting the radius of each extra dimension. Using the dimensional regularization technique, the Casimir energy reads,

$$\mathcal{E} = \int d^{D-1}p \sum_{n_1, n_2 = -\infty}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} \right]^{-s} \quad (17)$$

where in the end  $s = -1/2$  and  $D = 4$  as previously. The Casimir energy can be written in the form,

$$\mathcal{E} = \mathcal{E}^{D-1} + \mathcal{E}_{z_1, z_2} + \mathcal{E}_{z_1, z_2}(n_1) + \mathcal{E}_{z_1, z_2}(n_2) \quad (18)$$

with  $\mathcal{E}_{z_1, z_2}$ ,  $\mathcal{E}_{z_1, z_2}(n_1)$  and  $\mathcal{E}_{z_1, z_2}(n_2)$  the extra dimensional contributions to the Casimir energy, which will be the subject of our study in this section. The first term is equal to,

$$\mathcal{E}_{z_1, z_2} = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_1, n_2 = -\infty}^{\infty'} \left[ q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} \right]^{-(s - \frac{D-1}{2})} \quad (19)$$

and the other two,

$$\mathcal{E}_{z_1, z_2}(n_1) = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_1 = -\infty}^{\infty'} \left[ q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} \right]^{-(s - \frac{D-1}{2})} \quad (20)$$

and

$$\mathcal{E}_{z_1, z_2}(n_2) = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_2 = -\infty}^{\infty'} \left[ q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} \right]^{-(s - \frac{D-1}{2})} \quad (21)$$

The sums appearing in the most complicated contribution, namely equation (8), can be expressed in terms of a multidimensional zeta function, defined as,

$$\mathcal{M}_2(s_1; a_1, a_2; m_1, m_2) = \frac{1}{2} \sum_{n_1, n_2 = -\infty}^{\infty'} (a_1 n_1^{m_1} + a_2 n_2^{m_2})^{-s_1} \quad (22)$$

with  $m_1, m_2$  even positive integers. These and other more general multidimensional zeta functions were studied in the seminal paper of E. Elizalde [76]. Following [76], the multidimensional zeta function for our case, equals to,

$$\begin{aligned} \mathcal{M}_2(s_1; a_1, a_2; m_1, m_2) &\simeq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(s_1 + k)}{k! \Gamma(s_1)} \frac{(-a_1)^k}{a_2^{s_1+k}} \zeta(-m_1 k) \zeta(m_2(s_1 + k)) \\ &+ \frac{\Gamma(1/m_1) \Gamma(s_1 - 1/m_1)}{m_1 m_2^{s_1} \Gamma(s_1)} \left( \frac{a_2}{a_1} \right)^{1/m_1} \zeta(m_2(s_1 - 1/m_1)), \end{aligned} \quad (23)$$

Note that the above result is valid when  $m_i \geq 2$  and  $m_i$  is an even positive integer. There is an additional contribution to the zeta function which is negligible for  $m_i \geq 2$ . In our case the parameters are equal to:

$$\begin{aligned} a_1 &= q_1^{2(z_1-1)} \left( \frac{\pi}{R_1} \right)^{2z_1}, \quad a_2 = q_2^{2(z_2-1)} \left( \frac{\pi}{R_2} \right)^{2z_2} \\ m_1 &= 2z_1, \quad m_2 = 2z_2, \quad s_1 = s - \frac{D-1}{2} \end{aligned} \quad (24)$$

Hence the term  $\mathcal{E}_{z_1, z_2}$  can be written in terms of the multidimensional zeta function (23),

$$\mathcal{E}_{z_1, z_2} = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \mathcal{M}_2(s_1; a_1, a_2; m_1, m_2) \quad (25)$$

or equivalently, we have the expanded form:

$$\begin{aligned} \mathcal{E}_{z_1, z_2} &= \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{1}{2\Gamma(s)} \left( \sum_{k=0}^{\infty} \frac{\Gamma(s_1 + k)}{k!} \frac{(-a_1)^k}{a_2^{s_1+k}} \zeta(-m_1 k) \zeta(m_2(s_1 + k)) \right. \\ &\quad \left. + \frac{\Gamma(1/m_1) \Gamma(s_1 - 1/m_1)}{m_1 m_2^{s_1}} \left( \frac{a_2}{a_1} \right)^{1/m_1} \zeta(m_2(s_1 - 1/m_1)) \right) \end{aligned} \quad (26)$$

Moreover, the terms  $\mathcal{E}_{z_1, z_2}(n_1)$  and  $\mathcal{E}_{z_1, z_2}(n_2)$  can be written in terms of the Riemann zeta function (9) as we saw in the previous section,

$$\mathcal{E}_{z_1, z_2}(n_1) = \frac{2}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} q_1^{-2(z_1-1)(s - \frac{D-1}{2})} \left( \frac{\pi}{R_1} \right)^{-2z_1(s - \frac{D-1}{2})} \zeta\left(2z_1\left(s - \frac{D-1}{2}\right)\right) \quad (27)$$

$$\mathcal{E}_{z_1, z_2}(n_2) = \frac{2}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} q_2^{-2(z_2-1)(s - \frac{D-1}{2})} \left( \frac{\pi}{R_2} \right)^{-2z_2(s - \frac{D-1}{2})} \zeta\left(2z_2\left(s - \frac{D-1}{2}\right)\right)$$

### Singularities for Various $z_1, z_2$ Values

Let us present now the singularities that the extra dimensions related terms exhibit. The singularities of the terms  $\mathcal{E}_{z_1, z_2}(n_1)$  and  $\mathcal{E}_{z_1, z_2}(n_2)$  are similar to the singularities we analyzed in the previous section, so for  $z_1 \geq 2$  and  $z_2 \geq 2$  respectively, the aforementioned terms are singular. Particularly, we found that when  $z_1 \geq 2$  and  $z_2 \geq 2$ , the  $\mathcal{E}_{z_1, z_2}(n_1)$  and  $\mathcal{E}_{z_1, z_2}(n_2)$  terms contain singularities which are of the form:

$$\mathcal{E}_{z_1, z_2}(n_1) \sim \Gamma(-2), \quad \mathcal{E}_{z_1, z_2}(n_2) \sim \Gamma(-2) \quad (28)$$

The term  $\mathcal{E}_{z_1, z_2}$  contains various singularities for specific values of the summation parameter  $k$ , which depend on the Lifshitz parameters  $z_1$  and  $z_2$ . Remarkably, the singularity depends only on the value of  $z_2$ . The singularities are of the following type, and for the corresponding  $k$  values:

$$\begin{aligned} k &= 0, \quad \Gamma(-2) \\ k &= 1, \quad \Gamma(-1) \\ k &= 2, \quad \Gamma(0) \end{aligned} \quad (29)$$

Therefore the Lifshitz theory Casimir energy, contains additional singularities which are the singularities of the functions  $\mathcal{E}_{z_1, z_2}(n_1)$ ,  $\mathcal{E}_{z_1, z_2}$  and  $\mathcal{E}_{z_1, z_2}(n_2)$  for  $z_1, z_2 \geq 2$ . In turn, these singularities would need special treatment in order the result is credible, thus putting in question such extra dimensional spaces. Note that for  $z_2 = 1$  there is only one singularity and when  $z_1, z_2 \geq 2$  there are five sources of singularities. In the  $z_1 = z_2 = 1$  case, by applying the reflection formula (12), the contribution for  $z_2 = 1$  is recast into the following form:

$$\begin{aligned} \mathcal{E}_{z_1, z_2} = & \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{1}{2\Gamma(s)} \left( \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1-2(s_1+k)}{2})}{k!} \frac{(-a_1)^k}{a_2^{s_1+k}} \zeta(-m_1 k) \zeta(1-2(s_1+k)) \right) \\ & + \frac{\Gamma(1/m_1) \Gamma(s_1 - 1/m_1)}{m_1 m_2^{s_1}} \left( \frac{a_2}{a_1} \right)^{1/m_1} \zeta(m_2(s_1 - 1/m_1)) \end{aligned} \quad (30)$$

which is singular for  $k = 2$  since the zeta function has a singularity for that value of  $k$ . Note also that when  $z_2 = 1$ , and  $z_1 \geq 2$ , the only new singularity is contained in the term  $\mathcal{E}_{z_1, z_2}(n_1)$ , since all other terms contain singularities that the non-Lifshitz case has.

### 3 Casimir Energy for Massive Scalar Field $M_4 \times S_{z_1}^1$

In this section we consider the massive case of the scalar field in  $M_4 \times S_{z_1}^1$  spacetime, with the extra dimensions being of Lifshitz type. We adopt the conventions we used in section 1, regarding the scalings of the dimensions. In this case too the singularities depend on the  $z_1$  scaling parameter characterizing the extra dimension, as in the massless case. The total Casimir energy in the context of dimensional regularization reads,

$$\mathcal{E}^m = \int d^{D-1}p \sum_{n_1=-\infty}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + M^2 \right]^{-s} \quad (31)$$

We discriminate between the massive  $D-1$  contribution and the extra dimensional contribution,

$$\mathcal{E}^m = \mathcal{E}^{m, D-1} + \mathcal{E}_{z_1}^m \quad (32)$$

with  $\mathcal{E}^{m, D-1}$  the  $D-1$  contribution to the Casimir energy, and  $\mathcal{E}_{z_1}^m$  the extra dimensional contribution to the Casimir energy. These two are equal to:

$$\begin{aligned} \mathcal{E}^{m, D-1} &= \int d^{D-1}p \left[ \sum_{k=1}^{D-1} p_k^2 + M^2 \right]^{-s} \\ \mathcal{E}_{z_1}^m &= \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_1=-\infty}^{\infty'} \left[ q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + M^2 \right]^{-(s - \frac{D-1}{2})} \end{aligned} \quad (33)$$



The sum in the above expression can be written in terms of the multidimensional zeta function [76].

$$\begin{aligned}\mathcal{M}_1^c(s_1; a_1; m_1) &= \frac{1}{2} \sum_{n_1=-\infty}^{\infty'} (a_1 n_1^{m_1} + c)^{-s_1} \\ &\simeq \frac{1}{2 a_1^s \Gamma(s_1)} \sum_{p=0}^{\infty} \left(-\frac{c}{a_1}\right)^p \Gamma(s_1 + p) \zeta(m_1(s_1 + p))\end{aligned}\quad (34)$$

Hence, the extra dimension dependent part of the Casimir energy, namely  $\mathcal{E}_{z_1}^m$ , is written in the following form,

$$\mathcal{E}_{z_1}^m = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \mathcal{M}_1^c(s_1; a_1; m_1) \quad (35)$$

with  $s_1, m_1, a_1, c$  equal to:

$$\begin{aligned}a_1 &= q_1^{2(z_1-1)} \left(\frac{\pi}{R_1}\right)^{2z_1}, \quad s_1 = s - \frac{D-1}{2} \\ m_1 &= 2z_1, \quad c = M^2\end{aligned}\quad (36)$$

Relation (35) can be recast in the form:

$$\mathcal{E}_{z_1}^m = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{1}{\Gamma(s) 2 a_1^s} \sum_{p=0}^{\infty} \left(-\frac{c}{a_1}\right)^p \Gamma(s_1 + p) \zeta(m_1(s_1 + p)) \quad (37)$$

The singularities in relation (37) whenever  $z_1 \geq 2$  occur for three values of  $p$  and are of the form:

$$\begin{aligned}p = 0, \quad \mathcal{E}_{z_1}^m &\sim \Gamma(-2) \\ p = 1, \quad \mathcal{E}_{z_1}^m &\sim \Gamma(-1) \\ p = 2, \quad \mathcal{E}_{z_1}^m &\sim \Gamma(0)\end{aligned}\quad (38)$$

The  $z_1 = 1$  case renders the Casimir energy singular too, but contains only one singularity. This can easily be seen using the reflection formula for the zeta function (12), as we saw in the previous section. Hence the Lifshitz case Casimir energy contains more singularities, and hence should need much more regularization terms. Before closing this section, we present the massive scalar field case in the  $M_4 \times S_{z_1}^1 \times S_{z_2}^2$  spacetime. The total Casimir energy of the scalar field in this background, is of the form (in the end we take  $D = 4$  and  $s = -1/2$  as previously):

$$\mathcal{E}^m = \int d^{D-1}p \sum_{n_1, n_2=-\infty}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + q_1^{2(z_1-1)} \left(\frac{n_1 \pi}{R_1}\right)^{2z_1} + q_2^{2(z_2-1)} \left(\frac{n_2 \pi}{R_2}\right)^{2z_2} + M^2 \right]^{-s} \quad (39)$$

The above can be recast as:

$$\mathcal{E}^m = \mathcal{E}^{m, D-1} + \mathcal{E}_{z_1, z_2}^m + \mathcal{E}_{z_1, z_2}^m(n_1) + \mathcal{E}_{z_1, z_2}^m(n_2) \quad (40)$$

We can see that the Casimir energy contains four terms. The first one is the 3-dimensional Casimir energy,  $\mathcal{E}^{m,D-1}$  corresponding to our visible world spacetime, the second is  $\mathcal{E}_{z_1,z_2}^m$  which is the part that depends on both the two extra dimensional spaces and finally, the other two, namely  $\mathcal{E}_{z_1,z_2}^m(n_1)$  and  $\mathcal{E}_{z_1,z_2}^m(n_2)$  take into account the contributions of  $S_{z_1}^1$  and  $S_{z_2}^2$  correspondingly. The last two are equal to:

$$\begin{aligned}\mathcal{E}_{z_1,z_2}(n_2) &= \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_2=-\infty}^{\infty'} \left[ q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} + M^2 \right]^{-\left(s - \frac{D-1}{2}\right)} \\ \mathcal{E}_{z_1,z_2}(n_1) &= \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_1=-\infty}^{\infty'} \left[ q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + M^2 \right]^{-\left(s - \frac{D-1}{2}\right)}\end{aligned}\quad (41)$$

while the  $\mathcal{E}_{z_1,z_2}^m$  part is equal to:

$$\mathcal{E}_{z_1,z_2}^m = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n_1, n_2=-\infty}^{\infty'} \left[ q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} + M^2 \right]^{-\left(s - \frac{D-1}{2}\right)} \quad (42)$$

The singularities of the first two equations have been analyzed in the previous sections and we shall not pursue these issues further. We shall only give the generalized zeta function formula [76], in terms of which the term  $\mathcal{E}_{z_1,z_2}^m$  is expressed, which is,

$$\begin{aligned}\mathcal{M}_2^c(s_1; a_1, a_2; m_1, m_2) &= \frac{1}{2} \sum_{n_1, n_2=-\infty}^{\infty'} (a_1 n_1^{m_1} + a_2 n_2^{m_2} + c)^{-s_1} \\ &\simeq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(s_1 + k)}{k! \Gamma(s_1)} \frac{(-a_1)^k}{a_2^{s_1+k}} \zeta(-m_1 k) \mathcal{M}_1^{c/a_1}(s_1 + k; 1; m_2) \\ &\quad + \frac{\Gamma(1/m_1) \Gamma(s_1 - 1/m_1)}{m_1 a_2^{s_1} \Gamma(s_1)} \left( \frac{a_2}{a_1} \right)^{1/m_1} \mathcal{M}_1^{c/a_2}(s_1 - 1/m_1; 1; m_2)\end{aligned}\quad (43)$$

Hence the Casimir energy  $\mathcal{E}_{z_1,z_2}^m$  can be written in the following form:

$$\begin{aligned}\mathcal{E}_{z_1,z_2}^m &= \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{1}{2\Gamma(s)} \left( \sum_{k=0}^{\infty} \frac{\Gamma(s_1 + k)}{k! \Gamma(s_1)} \frac{(-a_1)^k}{a_2^{s_1+k}} \zeta(-m_1 k) \mathcal{M}_1^{c/a_1}(s_1 + k; 1; m_2) \right. \\ &\quad \left. + \frac{\Gamma(1/m_1) \Gamma(s_1 - 1/m_1)}{m_1 a_2^{s_1} \Gamma(s_1)} \left( \frac{a_2}{a_1} \right)^{1/m_1} \mathcal{M}_1^{c/a_2}(s_1 - 1/m_1; 1; m_2) \right)\end{aligned}\quad (44)$$

with:

$$\begin{aligned}a_1 &= q_1^{2(z_1-1)} \left( \frac{\pi}{R_1} \right)^{2z_1}, \quad a_2 = q_2^{2(z_2-1)} \left( \frac{\pi}{R_2} \right)^{2z_2} \\ m_1 &= 2z_1, \quad m_2 = 2z_2, \quad s_1 = s - \frac{D-1}{2}\end{aligned}\quad (45)$$

All the singularities that appear in the above expression have also been analyzed earlier, so we will not present them here.

## 4 Casimir Energy for the Massive Scalar Field for multi-Extra Dimensional Space

In this section we present the general case in which the extra dimensional space is of the form  $S_1^{z_1} \times S_1^{z_2} \times \dots \times S_1^{z_N}$ . The action of the free massive scalar field with  $N$ -periodic Lifshitz extra dimensions reads,

$$\mathcal{S} = \int dt d^{D-1}p \quad (46)$$

$$\times \int_0^{2\pi R_1} \int_0^{2\pi R_2} \dots \int_0^{2\pi R_N} dy_i \Phi^*(x_\mu, y_i) \left( \partial^\mu \partial_\mu - \sum_{i=1}^N q_i^{2(z_i-1)} (-\partial_{y_i}^2)^{z_i} + M^2 \right) \Phi(x_\mu, y_i)$$

with  $i = 1, \dots, N$ . Periodic boundary conditions are assumed for every compact extra dimension, that is,

$$\mathcal{E}_{z_1, z_2, \dots, z_N}^m = \int d^{D-1}p \quad (47)$$

$$\sum_{n_1, n_2, \dots, n_N = -\infty}^{\infty} \left[ \sum_{k=1}^{D-1} p_k^2 + q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} + \dots + q_N^{2(z_N-1)} \left( \frac{n_N \pi}{R_N} \right)^{2z_N} + M^2 \right]^{-s}$$

The total Casimir energy within the dimensional regularization context reads,

$$\mathcal{E}_{z_1, z_2, \dots, z_N}^c = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \times \quad (48)$$

$$\sum_{n_1, n_2, \dots, n_N = -\infty}^{\infty} \left[ q_1^{2(z_1-1)} \left( \frac{n_1 \pi}{R_1} \right)^{2z_1} + q_2^{2(z_2-1)} \left( \frac{n_2 \pi}{R_2} \right)^{2z_2} + \dots + q_N^{2(z_N-1)} \left( \frac{n_N \pi}{R_N} \right)^{2z_N} + M^2 \right]^{-(s - \frac{D-1}{2})}.$$

The above expression can be written in terms of the multidimensional zeta function [76],

$$\mathcal{M}_N^c(s_1; a_1, a_2, \dots, a_N; m_1, m_2, \dots, m_N) = \quad (49)$$

$$\frac{1}{2} \sum_{n_1, n_2, \dots, n_N = -\infty}^{\infty} (a_1 n_1^{m_1} + a_2 n_2^{m_2} + \dots + a_N n_N^{m_N} + c)^{-s_1}$$

$$\simeq \frac{1}{2 a_N \Gamma(s_1)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^p \frac{b_{i_r}^{-1/a_{i_r}}}{a_{i_r}} \Gamma\left(\frac{1}{a_{i_r}}\right) \sum_{k_{j_1}, \dots, k_{j_{N-p-1}}=0} \Gamma\left(s_1 + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^p \frac{1}{a_{i_r}}\right)$$

$$\times \prod_{l=1}^{N-p-1} \frac{(-1)^{k_{j_l}}}{k_{j_l}!} \zeta(-a_{j_l} k_{j_l}) M_1^{c/a_{i_r}} m_N \left( m_N \left( s_1 + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^p \frac{1}{a_{i_r}} \right); 1; m_N \right);$$

Therefore we can easily analyze the singularities for any number of extra dimensions. The formulas above can be generalized to include finite temperature corrections, but we defer this study to a future work.

## 5 Casimir Energy for Scalar Field with Three Spatial Lifshitz Dimensions

### Massless Scalar Case-Identical Scaling of the Space Dimensions

We shall now study the Casimir energy for a scalar field in three dimensional spacetime but with the space and time scaled differently. Particularly, we assume that the mass dimension of the spacetime coordinates is:

$$[t] = -z, \quad [x_i] = -1 \quad (50)$$

with  $i = 1, 2, 3$ . This case has been studied thoroughly and appears quite frequently in the literature, see for example [85]. Relation (50) stems from the Lifshitz scaling of the time coordinate in reference to that of the three spatial coordinates, namely:

$$x_i \rightarrow bx_i, \quad t \rightarrow b^z t \quad (51)$$

As we mentioned in the introduction, although such a scaling is different in spirit regarding the original idea of Lifshitz [100], the renormalization properties of such scalar theories are remarkable rendering such theories really valuable, although Lorentz violating. We want to compute the Casimir energy in the case the system is confined in a box, with lengths,  $0 \leq x_i \leq L_i$  and  $i = 1, 2, 3$ . Assuming relation (50), the action of the massless scalar field reads,,

$$\begin{aligned} \mathcal{S} = & \int dt \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} dx_1 dx_2 dx_3 \\ & \times \Phi^*(t, x_i) \left( (\partial^t)^2 - (-\partial_{x_1}^2)^z - (-\partial_{x_2}^2)^z - (-\partial_{x_3}^2)^z \right) \Phi(t, x_i) \end{aligned} \quad (52)$$

The scalar field obeys Dirichlet boundary conditions at the boundary of the box,

$$\begin{aligned} \Phi(x_1, x_2, 0) &= \Phi(x_1, x_2, L_3) = 0 \\ \Phi(x_1, 0, x_3) &= \Phi(x_1, L_2, x_3) = 0 \\ \Phi(0, x_2, x_3) &= \Phi(L_1, x_2, x_3) = 0 \end{aligned} \quad (53)$$

and as a consequence the Casimir energy is equal to:

$$\mathcal{E}_{z_1, z_2, z_3} = \sum_{n_1, n_2, n_3=1}^{\infty} \left[ \left( \frac{2n_1\pi}{L_1} \right)^{2z} + \left( \frac{2n_2\pi}{L_2} \right)^{2z} + \left( \frac{2n_3\pi}{L_3} \right)^{2z} \right]^{-s} \quad (54)$$

The expression above can be written in terms of a multidimensional zeta function [76], and particularly in terms of the following:

$$\begin{aligned}
\mathcal{M}_3(s_1; a_1, a_2, a_3; m_1, m_2, m_3) &= \sum_{n_1, n_2, n_3=1}^{\infty} (a_1 n_1^{m_1} + a_2 n_2^{m_2} + a_3 n_3^{m_3})^{-s_1} \quad (55) \\
&\simeq \frac{1}{a_3^{s_1} \Gamma(s_1)} \left( \sum_{k_1, k_2=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \frac{(-b_2)^{k_2}}{k_2!} \right. \\
&\quad \times \Gamma(s_1 + k_1 + k_2) \zeta(-m_1 k_1) \zeta(-m_2 k_2) \zeta(m_3(s_1 + k_1 + k_2)) \Big) \\
&\quad + \frac{\Gamma(1/m_2)}{m_2 b_2^{1/m_2}} \sum_{k_1=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \Gamma(s_1 + k_1 - 1/m_2) \zeta(-m_1 k_1) \zeta(m_3(s_1 + k_1 - 1/m_2)) \\
&\quad + \frac{\Gamma(1/m_1)}{m_1 b_1^{1/m_1}} \sum_{k_2=0}^{\infty} \frac{(-b_2)^{k_2}}{k_2!} \Gamma(s_1 + k_2 - 1/m_1) \zeta(-m_2 k_2) \zeta(m_3(s_1 + k_2 - 1/m_1)) \\
&\quad + \frac{\Gamma(1/m_1) \Gamma(1/m_2)}{m_1 b_1^{1/m_1} m_2 b_2^{1/m_2}} \Gamma(s_1 - 1/m_1 - 1/m_2) \zeta(m_3(s_1 - 1/m_1 - 1/m_2))
\end{aligned}$$

with  $b_j = a_j/a_3$ ,  $j = 1, 2$ . Thereby, the Casimir energy equals to:

$$\mathcal{E}_{z_1, z_2, z_3} = \mathcal{M}_3(s_1; a_1, a_2, a_3; m_1, m_2, m_3) \quad (56)$$

with the obvious identification,

$$\begin{aligned}
a_1 &= \left( \frac{\pi}{L_1} \right)^{2z}, \quad a_2 = \left( \frac{\pi}{L_2} \right)^{2z}, \quad a_3 = \left( \frac{\pi}{L_3} \right)^{2z} \quad (57) \\
m_1 &= m_2 = m_3 = 2z, \quad s_1 = -\frac{1}{2},
\end{aligned}$$

Before starting examining the singularities of the Lifshitz Casimir energy, let us point that the non-Lifshitz case ( $z=1$ ) contains only two singularities (poles of the gamma function and zeta function). Let us examine the singularities of the Casimir energy, for  $m_1 = m_2 = m_3 = 4$ , which corresponds to  $z = 2$ . In equation (55) the third line term is always regular for all  $z$ . For  $z = 2$ , the fourth and fifth line terms are singular, like in the  $z = 1$  case. In addition, the last line term is regular for  $z = 1$ , singular for  $z = 2$  and regular for  $z \geq 3$ . Remarkably, when  $z \geq 3$ , all the terms are regular. Hence we could say that the most problematic critical exponent for the case where an anisotropic scaling between space and time coordinates is deployed, is the one that corresponds to  $z = 2$ , at least when Casimir energy calculations are considered. However when  $z \geq 3$ , the Casimir energy is finite, free of singularities. This result is kind of unexpected, but we could reason that since the renormalization properties of Lifshitz type theories are better in reference to the non-Lifshitz ones, this fact is materialized in the Casimir energy too.

## Massless Scalar Case-non Identical Scaling of the Space Dimensions

We now address the problem in which the four dimensional spacetime has three spatial dimensions that are Lifshitz and one temporal dimension which is non-Lifshitz. The

dimensions are scaled in the following way:

$$x_i \rightarrow b^{z_i} x_i, \quad t \rightarrow bt \quad (58)$$

with  $i = 1, 2, 3$ . We assumed that the critical exponents of the spatial dimensions scalings are different. Therefore, the mass dimensions of the spacetime dimensions are:

$$[t] = -1, \quad [x_1] = -z_1, \quad [x_2] = -z_2, \quad [x_3] = -z_3 \quad (59)$$

The corresponding action of the free massless scalar field in this spacetime is,

$$\begin{aligned} \mathcal{S} = & \int dt \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} dx_1 dx_2 dx_3 \\ & \times \Phi^*(t, x_i) \left( (\partial^t)^2 - q_1^{2(z_1-1)} (-\partial_{x_2}^2)^{z_2} - q_2^{2(z_2-1)} (-\partial_{x_2}^2)^{z_2} - q_3^{2(z_3-1)} (-\partial_{x_3}^2)^{z_3} \right) \Phi(t, x_i) \end{aligned} \quad (60)$$

with  $i = 1, 2, 3$ . We confine the scalar field in a box, like in the previous section, and we impose Dirichlet boundary conditions at the boundaries of the three dimensional box. These are,

$$\begin{aligned} \Phi(x_1, x_2, 0) &= \Phi(x_1, x_2, L_3) = 0 \\ \Phi(x_1, 0, x_3) &= \Phi(x_1, L_2, x_3) = 0 \\ \Phi(0, x_2, x_3) &= \Phi(L_1, x_2, x_3) = 0 \end{aligned} \quad (61)$$

Accordingly, the Casimir energy reads,

$$\mathcal{E}_{z_1, z_2, z_3}^A = \sum_{n_1, n_2, n_3=1}^{\infty} \left[ q_1^{2(z_1-1)} \left( \frac{2n_1\pi}{L_1} \right)^{2z_1} + q_2^{2(z_2-1)} \left( \frac{2n_2\pi}{L_2} \right)^{2z_2} + q_3^{2(z_3-1)} \left( \frac{2n_3\pi}{L_3} \right)^{2z_3} \right]^{-s} \quad (62)$$

and using the multidimensional zeta function (55), it can be written as follows,

$$\mathcal{E}_{z_1, z_2, z_3}^A = \mathcal{M}_3(s_1; a_1, a_2, a_3; m_1, m_2, m_3) \quad (63)$$

with the parameters being equal to,

$$\begin{aligned} a_1 &= q_1^{2(z_1-1)} \left( \frac{\pi}{L_1} \right)^{2z_1}, \quad a_2 = q_2^{2(z_2-1)} \left( \frac{\pi}{L_2} \right)^{2z_2}, \quad a_3 = q_3^{2(z_3-1)} \left( \frac{\pi}{L_3} \right)^{2z_3} \\ m_1 &= 2z_1, \quad m_2 = 2z_2, \quad m_3 = 2z_3, \quad s_1 = -\frac{1}{2} \end{aligned} \quad (64)$$

Let us exploit the singularities in the present case, by exploring equation (55). To start with, consider the term in the third line, which is always regular (recall  $s_1 = -1/2$ ). We focus our interest to the terms appearing in the fourth, fifth and sixth line of equation (55). For  $z_1 = z_2 = 1$  and  $z_3 \geq 1$ , the fourth and fifth term are singular (gamma function poles) while the sixth is regular. In addition, for  $z_3 \geq 3$  and  $z_1 \geq 2$  and  $z_2 = 1$ , the fourth term is singular, the fifth term is regular and the sixth is regular. Consider now the case with  $z_3 \geq 1$  and  $z_1 = z_2 = 2$ . The fourth term is regular, the fifth term is regular and the sixth

is singular thus the number of singularities in the Lifshitz case (1 pole) is less than the one corresponding to the non-Lifshitz case (two poles). Finally, consider the case  $z_1, z_2 > 2$  and  $z_3 \geq 1$ , which is the most interesting one, since the Casimir energy is rendered finite, since all the terms are regular, in contrast to the non-Lifshitz one which is singular. This is the most interesting result for this situation, probably originating from the fact that Lifshitz theories are renormalized much more easily compared to the non-Lifshitz ones.

## Conclusions

In this paper we studied the singularities that the Casimir energy of a scalar field has, when Lifshitz dimensions are encountered. Particularly we were interested in revealing the situations in which the Casimir energy with Lifshitz dimensions has more singularities in reference to the non-Lifshitz one, and also how the singularities depend on the critical exponents of the Lifshitz dimensions. We addressed the problem in various spacetimes. Firstly we studied the massless scalar field in Minkowski spacetime with one and two circular extra dimensions, with the scalar field obeying periodic boundary conditions in the extra dimensions. In the case of one extra dimension we separated the 4 dimensional part of the Casimir energy from the extra dimensional contribution to the energy. We found that when the critical exponent is  $z_1 \geq 2$ , the Casimir energy is singular, a feature that the non-Lifshitz Casimir energy does not have. In the case of two extra dimensions, we found similar results. Specifically, when the critical exponents of each dimension separately satisfy  $z_i \geq 2$ , the massless scalar field Casimir energy is singular with the total number of singularities being larger than the corresponding non-Lifshitz one. We also studied the massive counterpart of the above two situations, where the singularities for  $z_1 \geq 2$  are much more severe, compared to the massless case. In addition we studied the massless scalar field Casimir energy in four dimensional spacetime, in which time scales differently in comparison to space dimensions. This case is very well studied in the literature. We confined the scalar field in a box, and having applied Dirichlet boundary conditions at the boundaries of the box, we found that the only problematic situation arises for  $z = 2$ . When  $z \geq 3$ , the total Casimir energy is finite, a result that probably stems from the renormalization properties of the Lifshitz quantum field theories. The drawback of the Lifshitz theories is the explicit breaking of Lorentz invariance, but in the case where boundaries are introduced, there is no room for Lorentz invariance anyway. Hence, these could prove valuable in field theoretic frameworks with boundaries. Finally we studied the massless scalar field confined in a box, but this time with the space dimensions having different critical exponents and obviously, the Lifshitz dimensions are the three space dimensions. In this case, the results are similar as in the previous case, that is, when  $z \geq 3$ , the Casimir energy is finite. Now why this number plays such an important role in the final result is something that should be thoroughly investigated.

Since the Casimir energy is an important feature of a consistent quantum field theory, being a manifestation of the vacuum of the theory, the study of the singularities that exhibits is of particular importance. This is owing to the fact that any infinite Casimir energy term must be properly regularized in a physically consistent way. Therefore, within

the theoretical framework of Lifshitz spacetime dimensions with the total number of dimensions being equal to four, one has to regularize equal or less infinite terms in order to properly regularize the Casimir energy in reference to the non-Lifshitz one. Moreover, this regularization must have a consistent underlying physical origin. But the fact that there exist critical exponent values for which the Casimir energy is free of singularities in four dimensional spacetime, gives the Lifshitz theories an intrinsic appeal. However, the Lifshitz extra dimensions case has more or equal singularities, compared to the non-Lifshitz case, which is a bad feature for any quantum field theory owing to the fact that we have to give sufficient explanations regarding their origin.

It would be very interesting to extend the above analysis in the finite temperature cases, or studying the fermionic Casimir energy at zero and finite temperature, with or without extra dimensions. Particularly, a natural question to ask is how the bag boundary conditions affect the Casimir energy within the theoretical framework of Lifshitz dimensions. Of equal importance is the electromagnetic field study of the Casimir energy with Lifshitz dimensions, and how gauges modify the result. We hope to address these problems in the future.

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